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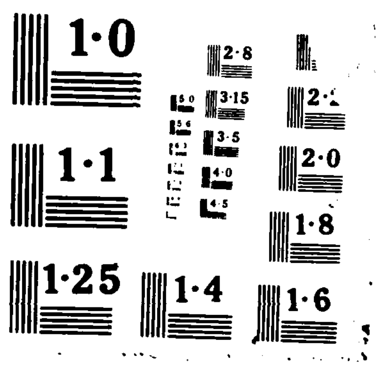
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CROSS STREAM DIFFERENCING FOR INTEGRAL BOUNDARY LAYER EQUATIONS

David Hally

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Approved by L.J. Leggat Director/Technology Division

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Abstract

An efficient, stable, explicit, first-order cross-stream differencing scheme having low truncation error is derived and applied to three dimensional integral boundary layer equations. The analysis is considered in detail for the particular case of the momentum integral equations and Head's entrainment equation with power law profiles and Mager's cross flow assumption. In comparison with upwinding differencing, the new scheme has better stability properties, smaller truncation error, and smaller artificial viscosity.

Résumé

Un schéma de différentiation transversale de premier ordre explicite, stable et efficace, avec une faible erreur de troncature, est établi et appliqué aux équations intégrales tridimensionnelles de la couche limite. L'analyse est considérée en détail dans le cas particulier des équations intégrales de quantité de mouvement et de l'équation d'entrainement de Head, avec des gradients exponentiels et l'hypothèse d'écoulement transversal de Mager. Comparé à la différentiation longitudinale, le nouveau schéma possède une plus grande stabilité, une plus petite erreur de troncature et une viscosité artificielle plus faible.

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Notation

$\mathbf{A}, \mathbf{B}$: Matrices of a hyperbolic system of first order partial differential equations.

$\mathbf{C}$: Matrix defined in equation (4.17).

$\mathbf{D}$: Matrix defined in equation (3.11).

D^+, D^- : Forward and backward differencing operators defined by equation (2.4).

$\mathbf{E}$: Matrix defined in equation (4.17).

E_t : Truncation error.

f : Independent variable in the convection equation.

f_k : Independent variables in the multi-dimensional hyperbolic system of equations.

$F^{k,j}$: Fourier component of f at the point (x_k, y_j).

$\mathbf{F}$: Matrix defined in equation (4.16).

$\mathbf{G}$: Matrix equal to $\mathbf{EC}^{-1}$.

$h_{11}, h_{12}, h_{21}, h_{22}, k_1, k_2$: Boundary layer velocity profile functions.

H : Shape factor.

P : Stability factor definined by equation (2.7).

$Q^1, Q^2, Q^{\perp 1}$: Independent variables in the boundary layer equations.

$Q^{\perp 2}, E^1, E^2$: Variables of which cross-stream derivatives are taken in the boundary layer equations.

t : Tan of the cross flow angle.

$\mathbf{T}$: Matrix defined in equation (4.16).

$\mathbf{v}^{(n)}$: The n^{th} eigenvector of $\mathbf{A}^{-1}\mathbf{B}$.

V^x : Velocity in the x direction.

V^y : Velocity in the y direction.

V^1, V^2 : Contravariant components of the potential flow velocity.

$V^{\perp 1}, V^{\perp 2}$: Contravariant components of a vector perpendicular to the potential flow velocity but having the same magnitude.

$\mathbf{w}^{(n)}$: Vectors defined by equation (3.5).

x, s : Non-orthogonal surface coordinates in which the boundary layer equations are solved.

x, y : Coordinates in which the convection equation is solved.

$\Delta s, \Delta x, \Delta y$: Grid spacing in the s, x, or y directions.

α : Coefficient defining the cross-stream differencing scheme.

β_k : Coefficients giving f_n as a linear combination of $v_k^{(n)}$.

δ : Boundary layer thickness.

δ_{mn} : Kronecker delta.

λ_n : The n^{th} eigenvalue of $\mathbf{A}^{-1}\mathbf{B}$.

μ : A parameter defined in equation (2.12).

Bold face characters are reserved for use as vectors or matrices.

1 Introduction

Numerical integration of a system of boundary layer equations is widely used for the prediction of the flow over ship hulls and aircraft. When the boundary layer equations are written in integral form and solved in a coordinate system roughly aligned with the flow, a popular method of discretization is a first order explicit scheme using upwind differencing for the cross-stream derivatives. This method has the virtue of extreme simplicity; however, a disadvantage of the method is that it is unstable when the direction of forward integration lies between the potential flow streamlines and the limiting streamlines at the hull surface. While the occurrence of instability will normally be infrequent and will usually not persist, it is most likely to occur when the cross-flow is high: that is, at a time when the modelling of the boundary layer is most suspect. When testing the validity of a boundary layer prediction method, it can be difficult to ascertain whether problems occurring at regions of high cross-flow are due to deficiencies in the boundary layer modelling or due to instabilities in the integration procedure. A completely stable differencing scheme would remove this ambiguity.

A second disadvantage of upwind differencing is that it is discontinuous with respect to the direction of the flow: i.e. an infinitesimal change in the flow which causes the cross-stream component of the flow to change sign, will cause finite changes in the solution due to the change in the direction of cross-stream differencing. If the boundary layer equations are being used in an iterative loop to account for the interaction between the boundary layer and the potential flow (see, for example, Chapter 8.3 in Reference [1]), this discontinuity can inhibit convergence.

In this memorandum an improved scheme for the cross-stream derivatives is proposed. The new scheme is also of first order, is stable for sufficiently small step size, and is continuous with respect to the direction of flow. It also has smaller truncation error and artificial viscosity than the upwind differencing scheme.

Motivation for the new differencing scheme for the boundary layer equations is given by first examining the pure convection equation with one dependent variable. For this equation, the new scheme is shown to be the optimal explicit differencing scheme in that it is (conditionally) stable and minimizes the truncation error. The scheme is then extended to systems of equations (i.e. equations with more than one dependent variable). Finally, use of the scheme in the context of the boundary layer equations is discussed. For this purpose the boundary layer equations are assumed to be the two momentum integral equations and Head's entrainment equation; these are the equations used by the HLLFLO computer programs developed at DREA for the prediction of the flow around ship hulls[1]. A complete discussion of these equations is given by Hally[2].

2 Stability of the Convection Equation

Consider, first, the simple one dimensional convection equation,

$$V^x \frac{\partial f}{\partial x} + V^y \frac{\partial f}{\partial y} = 0 \tag{2.1}$$

where V^x will be assumed positive with $V^x > |V^y|$. To solve the equation numerically, a rectangular grid (x_k, y_j) is used. For simplicity, the cross-stream step size will be assumed uniform: i.e. $y_{j+1} - y_j = \Delta y$ for every j. The derivatives are approximated by

$$\frac{\partial f}{\partial x} \approx \frac{f^{k+1,j} - f^{k,j}}{\Delta x} \tag{2.2}$$

$$\frac{\partial f}{\partial y} \approx \frac{1-\alpha}{2} D^+ f + \frac{1+\alpha}{2} D^- f \tag{2.3}$$

where $\Delta x = x_{k+1} - x_k$ and

$$D^+ f = \frac{f^{k,j+1} - f^{k,j}}{\Delta y} \qquad D^- f = \frac{f^{k,j} - f^{k,j-1}}{\Delta y} \tag{2.4}$$

The upwind differencing scheme has $\alpha = \mathrm{sgn}(V^y)$. Central differencing, which has accuracy of second order in the y-derivatives, has $\alpha = 0$.

The stability of a differencing scheme is determined by supposing a solution of the form

$$f^{k,j} = F^k e^{ij\Delta y} \tag{2.5}$$

Substitution into equation (2.1) and using equations (2.2) and (2.3) yields

$$F^{k+1,j} = F^{k,j} \left[1 - \frac{P}{2}[(1-\alpha)(e^{i\Delta y} - 1) + (1+\alpha)(1 - e^{-i\Delta y})]\right] \tag{2.6}$$

where

$$P \equiv \frac{V^y \Delta x}{V^x \Delta y} \tag{2.7}$$

The scheme is stable if $|F^{k+1,j}| \le |F^{k,j}|$ which occurs when

$$\alpha P \le 1 \quad \text{and} \quad P(\alpha - P) \ge 0 \tag{2.8}$$

(see, for example, Peyret and Taylor[3]). Hence, there is stability only if the scheme is weighted towards upwind differencing: i.e. $\mathrm{sgn}(\alpha) = \mathrm{sgn}(P)$. Complete upwind differencing is stable if $|P| \le 1$; this is called the Courant-Friedrichs-Lewy (CFL) condition. Central differencing is unstable.

The truncation error of the scheme of equations (2.2) and (2.3) is

$$E_t = V^x \frac{\Delta x}{2} \frac{\partial^2 f}{\partial x^2} - \alpha V^y \frac{\Delta y}{2} \frac{\partial^2 f}{\partial y^2} + O(\Delta x^2, \Delta y^2) \tag{2.9}$$

The term in Δy has a dissipative effect if $\alpha V^y > 0$. Since this term is physically similar to a diffusion term, it is often called "artificial viscosity". For a first order scheme the artificial viscosity is necessary for stability, but to avoid unwanted diffusive effects, it should be as small as possible. If V^x and V^y are constant, equations (2.1) and (2.7) can be used to rewrite equation (2.9) as

$$E_t = \frac{V^x \Delta y(\alpha - P)}{2} \frac{\partial^2 f}{\partial x \partial y} + O(\Delta x^2, \Delta y^2) \tag{2.10}$$

In order to minimize the truncation error, equation (2.10) suggests that one use $\alpha = P$, a scheme which will be termed partial upwinding for the purposes of this memorandum. This scheme is stable provided that $|P| \leq 1$, the CFL condition; it therefore requires no smaller step sizes than complete upwind differencing. Moreover, since α is continuous with respect to P, the differencing scheme is continuous with respect to V^y. The artifical viscosity is also reduced in comparison with that of the complete upwind differencing scheme.

In a more general convection equation, **A** and **B** may not be constant and there may be production terms.

$$A_{kn}(x, y, f) \frac{\partial f}{\partial x} + B_{kn}(x, y, f) \frac{\partial f}{\partial y} = c_k(x, y, f) \tag{2.11}$$

The differencing scheme defined by equations (2.2), (2.3), and $\alpha = P$ is stable when used for equation (2.11) but will have truncation error which is first order in Δx and Δy. However, if convection is the dominant mechanism describing the evolution of the solution, the truncation error will be small.

The relative accuracy of central differencing, upwind differencing, and partial upwind differencing have been compared using

$$\frac{V^y}{V^x} = -\frac{\mu y(1-y)}{1 + \mu x(1-2y)} \tag{2.12}$$

The solution to equation (2.1) is then

$$f(x, y) = f(0, y + \mu x y(1-y)) \tag{2.13}$$

In this flow, V^y is zero along the coordinate lines $y = 0$ and $y = 1$; hence, specification of the value of $f(0, y)$ for $0 \leq y \leq 1$ is sufficient to determine the value of f for any $x > 0$. In this sense, the flow mimics the evolution of the boundary layer on a ship hull which is confined between streamlines at the keel and at the waterline (see Hally[2]).

The parameter μ can be used to alter the relative magnitude of the components of the velocity in the x and y directions. Since one can show that

$$\left| \frac{\mu y(1-y)}{1 + \mu x(1-2y)} \right| < \frac{1 - \sqrt{1 - \mu^2}}{2|\mu|} \tag{2.14}$$

the maximum value for P, which will determine the stability of the differencing schemes, is

$$P_{max} = \frac{(1-\sqrt{1-\mu^2})\Delta x}{2|\mu|\Delta y} \tag{2.15}$$

Flows with larger μ will tend to be less stable.

The integration was performed for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Figure 2.1 shows the result of the integrations when $\Delta x = \Delta y = 0.05$, $\mu = 0.3$ and the starting values were given by $f(0,y) = 1 - \cos(\pi y)$. The deviation of the predicted values from the exact values are also shown. In this flow the velocity component V^y is both small and varies smoothly with y. Both upwind and partial upwind differencing are stable ($P_{max} = 0.077$) but the instability of the central differncing scheme is not clearly manifested. On the other hand, it is clear that the upwind differencing scheme has much higher truncation error relative to the other schemes.

Figure 2.2 shows the results of integrations with $\Delta x = 0.1$, $\Delta y = 0.0333$, $\mu = 0.9$ and starting values given by the pyramid function $f(0,y) = (1-|1-2y|)/2$. Again, the upwind and partial upwind differencing schemes are stable ($P_{max} = 0.940$), but in this case the instability of the central differencing scheme is clear. The rapid change in the y-derivative of V^y causes oscillations to appear in the central difference solution. Once more, the accuracy of the partial upwind scheme is the best.

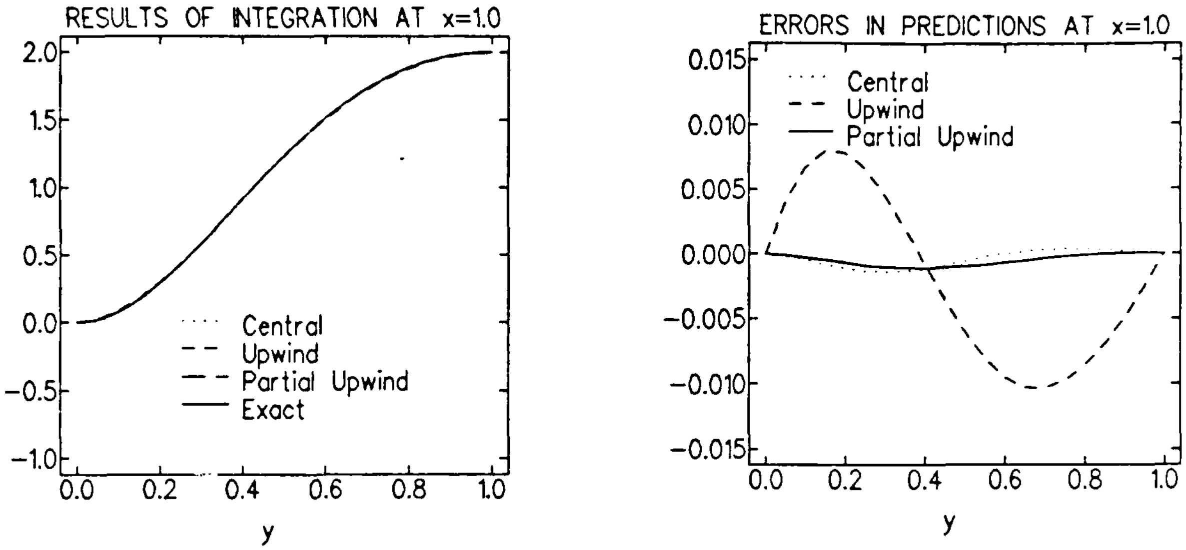

Figure 2.1: Results of integration with $f(0, y) = 1 - \cos(\pi y)$

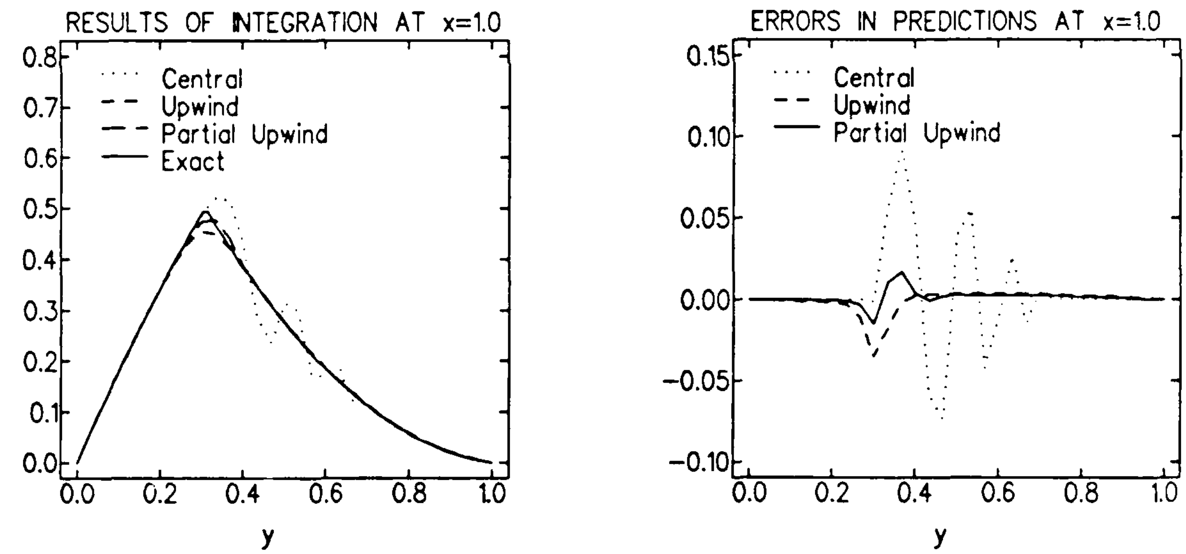

Figure 2.2: Results of integration with $f(0, y) = (1 - |1 - y|)/2$

3 Stability of a Hyperbolic System of Equations

The stability of a system of first order equations can often be reduced to the stability of a number of one dimensional equations. Suppose $f_n, n = 1, \ldots, N$ are variables dependent on x and y, and which satisfy

$$\sum_{n=1}^{N} A_{kn} \frac{\partial f_n}{\partial x} + B_{kn} \frac{\partial f_n}{\partial y} = 0 \tag{3.1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are $N \times N$ matrices and $\mathbf{A}$ is non-singular. Let λ_n be the eigenvalues and $\mathbf{v}^{(n)}$ be N linearly independent eigenvectors of $\mathbf{A}^{-1}\mathbf{B}$: i.e.

$$\sum_{j=1}^{N} \sum_{k=1}^{N} A_{mk}^{-1} B_{kj} v_j^{(n)} = \lambda_n v_m^{(n)} \tag{3.2}$$

or equivalently,

$$\sum_{j=1}^{N} (\lambda_n A_{kj} - B_{kj}) v_j^{(n)} = 0 \tag{3.3}$$

If all N eigenvalues are real and non-zero, equation (3.1) is said to be hyperbolic. If they are all real but some are zero, equation (3.1) is said to be mixed parabolic-hyperbolic. Henceforth each λ_n will be assumed to be real.

The dependent variables, f_n, can be expressed as a linear combination of the eigenvectors $\mathbf{v}^{(n)}$:

$$f_n = \sum_{k=1}^{N} \beta_k v_n^{(k)} \tag{3.4}$$

Defining vectors $\mathbf{w}^{(n)}$ such that

$$\sum_{k=1}^{N} w_k^{(n)} v_k^{(j)} = \delta_{nj} \tag{3.5}$$

one can represent the β_k explicitly:

$$\beta_k = \sum_{n=1}^{N} f_n w_n^{(k)} \tag{3.6}$$

Note, too, that

$$\sum_{k=1}^{N} w_n^{(k)} v_m^{(k)} \lambda_k = \sum_{k=1}^{N} A_{nk}^{-1} B_{km} \tag{3.7}$$

as can be verified by substituting equation (3.7) into equation (3.2).

After mutiplication by $\mathbf{A}^{-1}$ and substitution for f_n, equation (3.1) can be rewritten

$$\sum_{n=1}^{N} v_k^{(n)} \frac{\partial \beta_n}{\partial x} + v_k^{(n)} \lambda_n \frac{\partial \beta_n}{\partial y} = 0 \tag{3.8}$$

whence, since the $\mathbf{v}^{(n)}$ are linearly independent,

$$\frac{\partial \beta_n}{\partial x} + \lambda_n \frac{\partial \beta_n}{\partial y} = 0 \tag{3.9}$$

Hence, by applying a stable one-dimensional differencing scheme to each of the β_n, one obtains a stable differencing scheme for the entire linear system. The differencing scheme for f_n is recovered using equation (3.4).

$$\begin{aligned}
\frac{\partial f_n}{\partial y} &\approx \sum_{m=1}^{N} v_n^{(m)} \left[\frac{1-\alpha_m}{2} D^+ \beta_m + \frac{1+\alpha_m}{2} D^- \beta_m \right] \\
&= \sum_{m=1}^{N} \sum_{r=1}^{N} v_n^{(m)} w_r^{(m)} \left[\frac{1-\alpha_m}{2} D^+ f_r + \frac{1+\alpha_m}{2} D^- f_r \right] \\
&= \frac{D^+ f_n + D^- f_n}{2} - \sum_{m=1}^{N} D_{nm} \frac{D^+ f_m - D^- f_m}{2}
\end{aligned} \tag{3.10}$$

where

$$D_{nm} \equiv \sum_{r=1}^{N} v_n^{(r)} w_m^{(r)} \alpha_r \tag{3.11}$$

If all the eigenvalues are of the same sign, it is possible to choose all the α_m to be equal and have a stable scheme. This has the advantage that the cross-stream differencing is the same for each equation in the system. A complete upwind scheme would have $\alpha_m = \mathrm{sgn}(\lambda_m)$ while a possible extension of the partial upwind scheme of the previous section would be $\alpha_m = \max(\lambda_1, \ldots, \lambda_N)\Delta x/\Delta y$. By using the maximum eigenvalue one can ensure stability by choosing Δx sufficiently small that $\alpha_m < 1$. When all the α_m are equal, equation (3.10) reduces to the simple counterpart of equation (2.3):

$$\frac{\partial f_n}{\partial y} \approx \frac{1-\alpha}{2} D^+ f_n + \frac{1+\alpha}{2} D^- f_n \tag{3.12}$$

However, if the eigenvalues are not all of the same sign, then to maintain at least partial upwinding to provide stability, some of the α_m must be positive and some must be negative. Thus, in general it is not possible to have complete stability while retaining the simplicity of having all α_m equal.

The logical extension of the partial upwinding scheme of the previous section, is to set $\alpha_m = P_m$. Then

$$D_{nm} = \frac{\Delta x}{\Delta y} \sum_{r=1}^{N} v_n^{(r)} w_m^{(r)} \lambda_r = \frac{\Delta x}{\Delta y} \sum_{k=1}^{N} A_{nk}^{-1} B_{km} \tag{3.13}$$

whence

$$\mathbf{D} = \mathbf{A}^{-1}\mathbf{B}\Delta x/\Delta y \tag{3.14}$$

The truncation error when $\mathbf{D}$ is defined in this way is of second order in Δx and Δy.

4 A Differencing Scheme for the Boundary Layer Equations

As mentioned in the introduction, the boundary layer equations discussed here are assumed to be of the form used in the DREA HLLFLO programs[1]. These are a system of three coupled first order equations: the two momentum integral equations and Head's entrainment equation. They can be written in the form[2],

$$\frac{\partial Q^1}{\partial x} + \frac{\partial Q^2}{\partial s} = \text{production terms} \tag{4.1}$$

$$\frac{\partial Q^{\perp 1}}{\partial x} + \frac{\partial Q^{\perp 2}}{\partial s} = \text{production terms} \tag{4.2}$$

$$\frac{\partial E^1}{\partial x} + \frac{\partial E^2}{\partial s} = \text{production terms} \tag{4.3}$$

where

$$Q^1 = \sqrt{g}\delta(V^1 h_{11} + tV^{\perp 1} h_{12}) \tag{4.4}$$

$$Q^2 = \sqrt{g}\delta(V^2 h_{11} + tV^{\perp 2} h_{12}) \tag{4.5}$$

$$Q^{\perp 1} = \sqrt{g}\delta t(V^1 h_{21} + tV^{\perp 1} h_{22}) \tag{4.6}$$

$$Q^{\perp 2} = \sqrt{g}\delta t(V^2 h_{21} + tV^{\perp 2} h_{22}) \tag{4.7}$$

$$E^1 = \sqrt{g}\delta(V^1 k_1 + tV^{\perp 1} k_2) \tag{4.8}$$

$$E^2 = \sqrt{g}\delta(V^2 k_1 + tV^{\perp 2} k_2) \tag{4.9}$$

δ is the boundary layer thickness, and t is the tan of the cross-flow angle, V^1 and V^2 are the contravariant components of the potential flow velocity, $V^{\perp 1}$ and $V^{\perp 2}$ are the contravariant components of a vector perpendicular to the potential flow velocity and having the same magnitude, x and s are non-orthogonal coordinates which run over the surface of flow, and g is the determinant of the metric tensor for the coordinates (x, s). The values of V^1, V^2, $V^{\perp 1}$, $V^{\perp 2}$, and g are known for any given x and s.

The profile functions h_{11}, h_{12}, h_{21}, h_{22}, k_1, and k_2 depend on the velocity profile assumptions made for the boundary layer calculation. In general, the profile functions may depend on δ, t, and a third independent variable usually taken to be the shape factor, H, though more commonly they depend only upon H and t or on H alone.

A simple variant of upwind differencing can be used to integrate the boundary layer equations (4.1) – (4.3). If V^2/V^1 and $(V^2 + tV^{\perp 2})/(V^1 + tV^{\perp 1})$ both have the same sign (i.e. both potential streamlines and limiting streamlines lie on the same side as the x coordinate lines), upwind differencing is used for each of Q^2, $Q^{\perp 2}$, and E^2. However, if they have different signs (the x coordinate line bisects the two streamlines), no unique upwind direction is identified: central differencing is then used. Unfortunately, as has been demonstrated above, central differencing is unstable. Moreover, it is clear from Section 3 that no simple scheme in which the same value for α is used for each of Q^2, $Q^{\perp 2}$, and E^2 can be stable when the x coordinate line lies between the characteristic directions defined by λ_1 and λ_3.

On the other hand, by changing independent variables to the boundary layer variables δ, H, and t, the boundary layer equations may be rewritten in the form of equation (2.11) with $f_1 = \delta$, $f_2 = H$, $f_3 = t$, and

$$\mathbf{A} = \begin{bmatrix} \frac{\partial Q^1}{\partial \delta} & \frac{\partial Q^1}{\partial H} & \frac{\partial Q^1}{\partial t} \\ \frac{\partial Q^{\perp 1}}{\partial \delta} & \frac{\partial Q^{\perp 1}}{\partial H} & \frac{\partial Q^{\perp 1}}{\partial t} \\ \frac{\partial E^1}{\partial \delta} & \frac{\partial E^1}{\partial H} & \frac{\partial E^1}{\partial t} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \frac{\partial Q^2}{\partial \delta} & \frac{\partial Q^2}{\partial H} & \frac{\partial Q^2}{\partial t} \\ \frac{\partial Q^{\perp 2}}{\partial \delta} & \frac{\partial Q^{\perp 2}}{\partial H} & \frac{\partial Q^{\perp 2}}{\partial t} \\ \frac{\partial E^2}{\partial \delta} & \frac{\partial E^2}{\partial H} & \frac{\partial E^2}{\partial t} \end{bmatrix} \tag{4.10}$$

A and **B** are easy to determine analytically and, since **A** is only 3×3, $\mathbf{A}^{-1}$ is also easily calculated. Hence, the differencing scheme described in the previous section may be calculated easily. It is well-known that the characteristics of the boundary layer equations lie between the potential flow streamlines and the limiting streamlines at the hull surface; this is equivalent to saying that the eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$ lie between V^2/V^1 and $(V^2 + tV^{\perp 2})/(V^1 + tV^{\perp 1})$. Hence, the CFL condition is satisfied provided that

$$\max\left(\frac{V^2 \Delta x}{V^1 \Delta s}, \frac{(V^2 + tV^{\perp 2})\Delta x}{(V^1 + tV^{\perp 1})\Delta s}\right) \leq 1 \tag{4.11}$$

The differencing scheme just described is with respect to the independent variables δ, H and t. To generate corresponding difference schemes for Q^2, $Q^{\perp 2}$ and E^2, the correspondence

$$\begin{bmatrix} \frac{\partial Q^2}{\partial s} \\ \frac{\partial Q^{\perp 2}}{\partial s} \\ \frac{\partial E^2}{\partial s} \end{bmatrix} = \mathbf{B} \begin{bmatrix} \frac{\partial \delta}{\partial s} \\ \frac{\partial H}{\partial s} \\ \frac{\partial t}{\partial s} \end{bmatrix} \tag{4.12}$$

is used. Equation (3.10) is then an appropriate difference scheme where

$$(f_1, f_2, f_3) = (Q^2, Q^{\perp 2}, E^2) \tag{4.13}$$

and

$$\mathbf{D} = \mathbf{B}(\mathbf{A}^{-1}\mathbf{B})\mathbf{B}^{-1}\Delta x/\Delta s = \mathbf{B}\mathbf{A}^{-1}\Delta x/\Delta s \tag{4.14}$$

If it is assumed that the profile functions depend only on H, further simplifications are possible. In this case one has

$$\mathbf{A} = \sqrt{g}\mathbf{T}(V^1\mathbf{C} + tV^{\perp 1}\mathbf{E})\mathbf{F} \quad ; \quad \mathbf{B} = \sqrt{g}\mathbf{T}(V^2\mathbf{C} + tV^{\perp 2}\mathbf{E})\mathbf{F} \tag{4.15}$$

where

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad \mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{bmatrix} \tag{4.16}$$

$$\mathbf{C} = \begin{bmatrix} h_{11} & h'_{11} & 0 \\ h_{21} & h'_{21} & h_{21} \\ k_1 & k'_1 & 0 \end{bmatrix} \quad , \quad \mathbf{E} = \begin{bmatrix} h_{12} & h'_{12} & h_{12} \\ h_{22} & h'_{22} & 2h_{22} \\ k_2 & k'_2 & k_2 \end{bmatrix} \tag{4.17}$$

The primes denote differentiation with respect to H. Substitution into equation (4.15) gives

$$\begin{aligned} \mathbf{BA}^{-1} &= \mathbf{T}(V^2\mathbf{C} + tV^{\perp 2}\mathbf{E})(V^1\mathbf{C} + tV^{\perp 1}\mathbf{E})^{-1}\mathbf{T}^{-1} \\ &= \mathbf{T}(V^2 + tV^{\perp 2}\mathbf{EC}^{-1})(V^1 + tV^{\perp 1}\mathbf{EC}^{-1})^{-1}\mathbf{T}^{-1} \end{aligned} \tag{4.18}$$

Since the x coordinate lines are assumed to lie roughly along the potential flow streamlines, and since t is usually (though not invariably) significantly smaller than 1.0, it is reasonable to assume that $tV^{\perp 1} << V^1$ and retain only first order terms in equation (4.18).

$$\begin{aligned} \mathbf{BA}^{-1} &= \mathbf{T}\left[\frac{V^2}{V^1} + \frac{V^1V^{\perp 2} - V^{\perp 1}V^2}{(V^1)^2}t\mathbf{G}\right]\mathbf{T}^{-1} \\ &= \mathbf{T}\left[\frac{V^2}{V^1} + \frac{t}{\sqrt{g}}\left(\frac{U}{V^1}\right)^2\mathbf{G}\right]\mathbf{T}^{-1} \end{aligned} \tag{4.19}$$

where

$$\mathbf{G} \equiv \mathbf{EC}^{-1} \tag{4.20}$$

and U is the magnitude of the potential flow velocity. The last equality in equation (4.19) follows from the relations between $\mathbf{V}$ and $\mathbf{V}^{\perp}$ (see Appendix A.5 in Hally[2]).

The matrix $\mathbf{G}$ depends only on H. Its components may be written explicitly in terms of the profile functions as follows

$$G_{11} = \frac{k_1(k'_2h_{21} - k_2h'_{21})}{h_{21}(h'_{11}k_1 - h_{11}k'_1)} \tag{4.21}$$

$$G_{12} = 1 + \frac{k_2}{h_{21}} \tag{4.22}$$

$$G_{13} = -\frac{h_{11}G_{11}}{k_1} \tag{4.23}$$

$$G_{21} = \frac{k_1(h'_{22}h_{21} - h_{22}h'_{21}) - h_{22}(h'_{21}k_1 - h_{21}k'_1)}{h_{21}(h'_{11}k_1 - h_{11}k'_1)} \tag{4.24}$$

$$G_{22} = \frac{2h_{22}}{h_{21}} \tag{4.25}$$

$$G_{23} = \frac{h_{22}(h'_{21}h_{11} - h_{21}h'_{11}) - h_{11}(h'_{22}h_{21} - h_{22}h'_{21})}{h_{21}(h'_{11}k_1 - h_{11}k'_1)} \tag{4.26}$$

$$G_{31} = G_{11} \tag{4.27}$$

$$G_{32} = G_{12} - 1 \tag{4.28}$$

$$G_{33} = G_{13} \tag{4.29}$$

where use has been made of the identity

$$h_{12} = h_{21} + k_2 \tag{4.30}$$

For the particular case of power law profiles with Mager's cross-flow assumption (see Hally[2]), the profile functions are

$$h_{11} = \frac{H-1}{H(H+1)} \tag{4.31}$$

$$h_{12} = h_{21} + k_2 \tag{4.32}$$

$$h_{21} = \frac{-2}{H(H+1)(H+2)} \tag{4.33}$$

$$h_{22} = \frac{-24}{H(H+1)(H+2)(H+3)(H+4)} \tag{4.34}$$

$$k_1 = \frac{2}{H+1} \tag{4.35}$$

$$k_2 = \frac{16}{(H+1)(H+3)(H+5)} \tag{4.36}$$

and the components of **G** are

$$G_{11} = G_{31} = \frac{96H(H^2+5H+5)}{(H+2)(H+3)^2(H+5)^2} \tag{4.37}$$

$$G_{12} = -\frac{(H-1)(7H+15)}{(H+3)(H+5)} \tag{4.38}$$

$$G_{13} = G_{33} = \frac{-48(H-1)(H^2+5H+5)}{(H+2)(H+3)^2(H+5)^2} \tag{4.39}$$

$$G_{21} = -\frac{24(5H^2+24H+24)}{(H+2)^2(H+3)^2(H+4)^2} \tag{4.40}$$

$$G_{22} = \frac{24}{(H+3)(H+4)} \tag{4.41}$$

$$G_{23} = \frac{24(3H^2+14H+13)}{(H+2)^2(H+3)^2(H+4)^2} \tag{4.42}$$

$$G_{32} = -\frac{8H(H+2)}{(H+3)(H+5)} \tag{4.43}$$

To a very good approximation, $\mathbf{G}$ is linear in H over the range of interest (H is usually in the range 1.2 to 1.5). A linear approximation for $\mathbf{G}$ was generated using a least squares fit for the range $H \in [1.2, 1.5]$, yielding

$$\mathbf{G} \approx \begin{bmatrix} 0.46785 & 0.84360 & 0.19553 \\ -0.56021 & 1.61323 & 0.31568 \\ 0.46786 & -0.15638 & 0.19553 \end{bmatrix} + H \begin{bmatrix} 0.16237 & -0.85387 & -0.21027 \\ 0.22268 & -0.43032 & -0.12418 \\ 0.16237 & -0.85389 & -0.21027 \end{bmatrix} \tag{4.44}$$

The components of $\mathbf{G}$ and the approximations to them are shown in Figure 4.1.

Substitution of the above approximations into equation (3.10) with the identifications in equation (4.13) yields the following approximation to the stable scheme with $\alpha_n = \lambda_n \Delta x / \Delta s$; therefore it should be stable for all reasonable values of the boundary layer parameters.

$$\begin{aligned} \frac{\partial (Q^2)}{\partial y} \approx\; & \frac{1}{2}\left(1 - \frac{V^2 \Delta x}{V^1 \Delta s}\right) D^+ Q^2 + \frac{1}{2}\left(1 + \frac{V^2 \Delta x}{V^1 \Delta s}\right) D^- Q^2 \\ & + \frac{1}{2\sqrt{g}}\left(\frac{U}{V^1}\right)^2 \Big[t G_{11}(D^+Q^2 - D^-Q^2) + G_{12}(D^+Q^{\perp 2} - D^-Q^{\perp 2}) \\ & + G_{13} t (D^+E^2 - D^-E^2) \Big] \end{aligned} \tag{4.45}$$

$$\begin{aligned} \frac{\partial Q^{\perp 2}}{\partial y} \approx\; & \frac{1}{2}\left(1 - \frac{V^2 \Delta x}{V^1 \Delta s}\right) D^+ Q^{\perp 2} + \frac{1}{2}\left(1 + \frac{V^2 \Delta x}{V^1 \Delta s}\right) D^- Q^{\perp 2} \\ & + \frac{t}{2\sqrt{g}}\left(\frac{U}{V^1}\right)^2 \Big[t G_{21}(D^+Q^2 - D^-Q^2) + G_{22}(D^+Q^{\perp 2} - D^-Q^{\perp 2}) \\ & + t G_{23}(D^+E^2 - D^-E^2) \Big] \end{aligned} \tag{4.46}$$

$$\begin{aligned} \frac{\partial E^2}{\partial y} \approx\; & \frac{1}{2}\left(1 - \frac{V^2 \Delta x}{V^1 \Delta s}\right) D^+ E^2 + \frac{1}{2}\left(1 + \frac{V^2 \Delta x}{V^1 \Delta s}\right) D^- E^2 \\ & + \frac{1}{2\sqrt{g}}\left(\frac{U}{V^1}\right)^2 \Big[t G_{31}(D^+Q^2 - D^-Q^2) + G_{32}(D^+Q^{\perp 2} - D^-Q^{\perp 2}) \\ & + t G_{33}(D^+E^2 - D^-E^2) \Big] \end{aligned} \tag{4.47}$$

This scheme is efficient to calculate (since all the boundary layer parameters are known), has small truncation error and small artifical viscosity, and is continuous with respect to all boundary layer and potential flow parameters.

When the cross-flow is large, when the shape factor lies outside the range [1.2,1.5], or when profiles other than the power law profiles are used, a stable differencing scheme can be generating by calculating $\mathbf{BA}^{-1}$ explicitly and using equations (3.10) and (4.14). While not as efficient as using equations (4.45) – (4.47), the computational effort is not high as $\mathbf{A}$ and $\mathbf{B}$ are only 3×3.

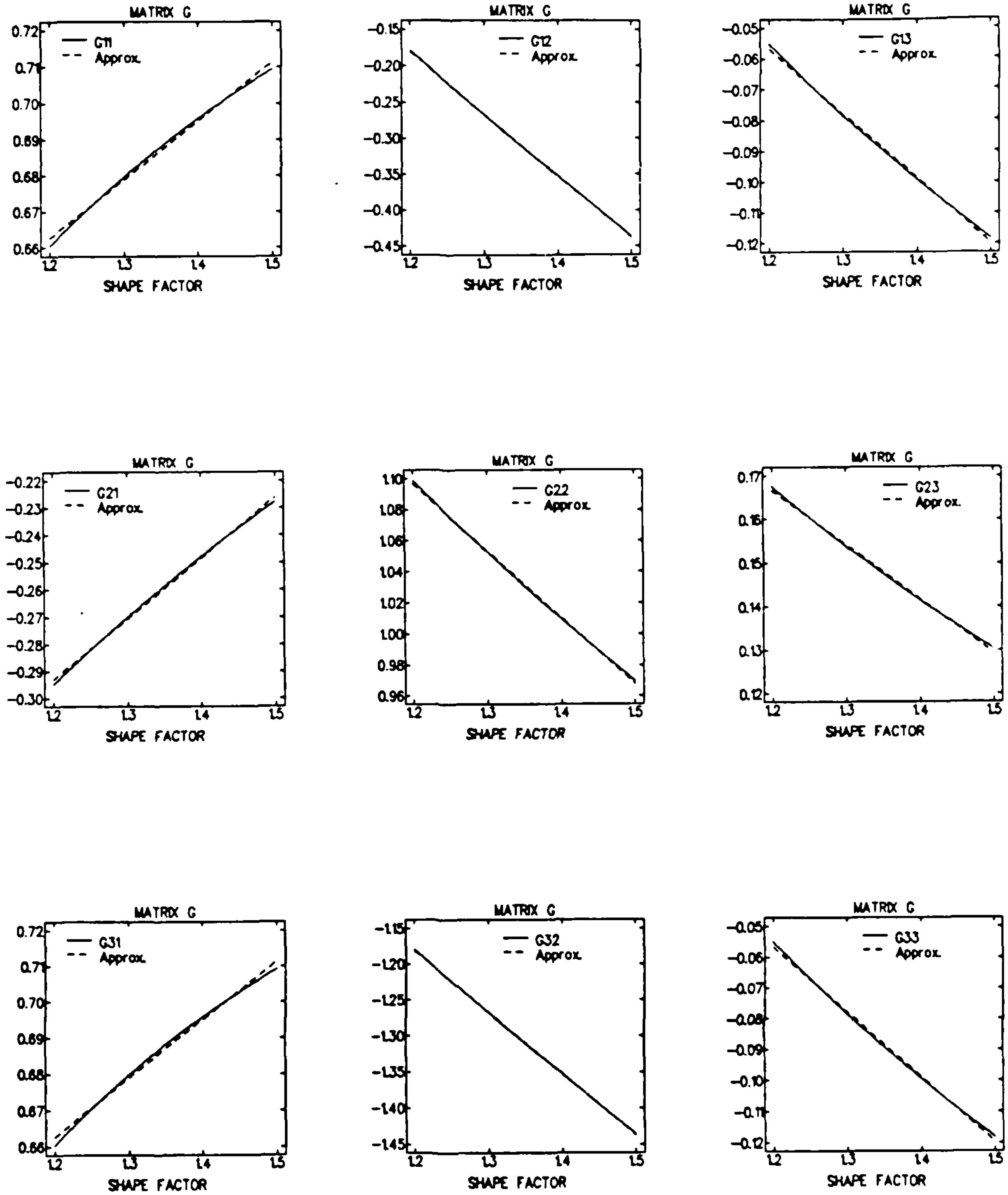

Figure 4.1: Components of **G** as functions of H

5 Concluding Remarks

A differencing scheme for the cross-stream derivatives of integral boundary layer equations has been discussed. In comparison with upwinding differencing, the new scheme has better stability properties, smaller truncation error, smaller artificial viscosity, and is continuous with respect to the direction of flow. Hence, it is a more accurate and more reliable method for the solution of the boundary layer equations.

When applied to the boundary layer equations used by the DREA HLLFLO programs, it has been shown that the improved differencing scheme can be written in a form which is straightforward and efficient to calculate.

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[3] R. Peyret and T. D. Taylor, *Computational Methods for Fluid Flow* (Springer Verlag, New York, 1983), Ch. 2.6.

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An efficient, stable, explicit, first-order cross-stream differencing scheme having low truncation error is derived and applied to three dimensional integral boundary layer equations. The analysis is considered in detail for the particular case of the momentum integral equations and Head's entrainment equation with power law profiles and Mager's cross flow assumption. In comparison with upwinding differencing, the new scheme has better stability properties, smaller truncation error, and smaller artificial viscosity.

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